

GRÖBNER BASES VIA LINKAGE

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ABSTRACT. In this paper, we give a sufficient condition for a set \mathcal{G} of polynomials to be a Gröbner basis with respect to a given term-order for the ideal I that it generates. Our criterion depends on the linkage pattern of the ideal I and of the ideal generated by the initial terms of the elements of \mathcal{G} . We then apply this criterion to ideals generated by minors and pfaffians. More precisely, we consider large families of ideals generated by minors or pfaffians in a matrix or a ladder, where the size of the minors or pfaffians is allowed to vary in different regions of the matrix or the ladder. We use the sufficient condition that we established to prove that the minors or pfaffians form a reduced Gröbner basis for the ideal that they generate, with respect to any diagonal or anti-diagonal term-order. We also show that the corresponding initial ideal is Cohen-Macaulay and squarefree, and that the simplicial complex associated to it is vertex decomposable, hence shellable. Our proof relies on known results in liaison theory, combined with a simple Hilbert function computation. In particular, our arguments are completely algebraic.

INTRODUCTION

Gröbner bases are the most widely applicable computational tool available in the context of commutative algebra and algebraic geometry. However they also are an important theoretical tool, as they can be used to establish properties such as, e.g., primality, normality, Cohen-Macaulayness, and to give formulas for the height of an ideal. Liaison theory, or linkage, on the other hand, is mostly regarded as a classification tool. In fact, much effort has been devoted in recent years to the study of liaison classes, in particular to deciding which ideals belong to the G-liaison class of a complete intersection. However, a clear understanding of the liaison pattern of an ideal often allows us to recursively compute invariants such as its Hilbert function and graded Betti numbers.

In this paper we introduce liaison-theoretic methods as a tool in the theory of Gröbner bases. More precisely, we deduce that a certain set \mathcal{G} of polynomials is a Gröbner basis for the ideal I that it generates by understanding the linkage pattern of the ideal I and of the monomial ideal generated by the initial terms of the elements of \mathcal{G} . Concretely,

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we apply this reasoning to ideals generated by minors or pfaffians, whose liaison pattern we understand.

Ideals generated by minors or pfaffians have been studied extensively by both commutative algebraists and algebraic geometers. The study of determinantal rings and varieties is an active area of research per se, but it has also been instrumental to the development of new techniques, which have become part of the commonly used tools in commutative algebra. Ideals generated by minors and pfaffians are studied in invariant theory and combinatorics, and are relevant in algebraic geometry. In fact, many classical varieties such as the Veronese and the Segre varieties are cut out by minors or pfaffians. Degeneracy loci of morphisms between direct sums of line bundles over projective space have a determinantal description, as do Schubert varieties, Fitting schemes, and some toric varieties. Ideals generated by minors or pfaffians are often investigated by commutative algebraists by means of Gröbner basis techniques (see, e.g., [1], [30], [20]). Using such tools, many families of ideals generated by minors or pfaffians have been shown to enjoy properties such as primality, normality, and Cohen-Macaulayness. A different approach to the study of ideals generated by minors is via flags and degeneracy loci, and was initiated in [12]. Such an approach allows one to establish that these ideals are normal, Cohen-Macaulay and have rational singularities (see, e.g., [22], [23], [25]).

In recent years, much progress has been made towards understanding determinantal ideals and varieties also from the point of view of liaison theory. A central open question in liaison theory asks whether every arithmetically Cohen-Macaulay scheme is *glicci*, i.e., whether it belongs to the G -liaison class of a complete intersection. In [13], [21], [15], [16], [17], [10], and [18], several families of ideals generated by minors or pfaffians are shown to be *glicci*. More precisely, it is shown that they can be obtained from an ideal generated by linear forms via a sequence of ascending elementary G -biliaisons. Moreover, each of the elementary G -biliaisons takes place between two ideals which both belong to the family in question. Since the linkage steps are described very explicitly, in theory it is possible to use the linkage results to recursively compute invariants or establish properties of these ideals. This has been done, e.g., in [7] using the linkage results from [16].

Rather than contributing to the theory of liaison (see for instance [27]), in this paper we give a new method of using liaison as a tool. More precisely, we consider large families of ideals generated by minors or pfaffians in a matrix or a ladder, namely pfaffian ideals of ladders, mixed determinantal, and symmetric mixed determinantal ideals. Combining the liaison results from [16], [10], and [18] with a Hilbert function computation, we are able to prove that the pfaffians or the minors are a reduced Gröbner basis for the ideal that they generate, with respect to any anti-diagonal or diagonal term-order. Moreover, we show the simplicial complex corresponding to the initial ideal of any ideal in the family that we consider is vertex decomposable. Vertex decomposability is a strong property, which in particular implies shellability of the complex and Cohen-Macaulayness of the associated initial ideal.

In Section 1, we prove a lemma which will be central to the subsequent arguments (Lemma 1.12). The lemma gives a sufficient criterion for a monomial ideal to be the initial ideal of a given ideal J . Both the ideal J and the “candidate” initial ideal are constructed

via Basic Double Linkage or elementary biliaison. In Section 2 we use Lemma 1.12 to prove that the maximal minors of a matrix of indeterminates are a Gröbner basis of the ideal that they generate with respect to any diagonal (or anti-diagonal) term-order. Although the result is well-known, we wish to illustrate our method by showing how it applies to this simple example. In Sections 3, 4, and 5, we apply our technique to ideals generated by: pfaffians of mixed size in a ladder of a skew-symmetric matrix of indeterminates, minors of mixed size in a ladder of a symmetric matrix of indeterminates, and minors of mixed size in a ladder of a matrix of indeterminates. We prove that the natural generators of these ideals are a Gröbner basis with respect to any diagonal (in the case of minors of a symmetric matrix) or anti-diagonal (in the case of pfaffians or minors in a generic matrix) term-order. We also prove that the corresponding initial ideals are Cohen-Macaulay, and that the associated simplicial complexes are vertex decomposable. While Sections 1 and 2 are meant to be read first, Sections 3, 4, and 5 can be read independently of each other, and in any order. In the appendix, we indicate how our liaison-theoretic approach can be made self-contained in order to derive also all the classical Gröbner basis results about ladder determinantal ideals from one-sided ladders.

1. LINKAGE AND GRÖBNER BASES

Let K be an arbitrary field and let R be a standard graded polynomial ring in finitely many indeterminates over K . In this section, we give a sufficient condition for a set \mathcal{G} of polynomials to be a Gröbner basis with respect to a given term-order for the ideal I that it generates. Our criterion depends on the linkage pattern of the ideal I and of the monomial ideal generated by the initial terms of the elements of \mathcal{G} .

In order to use geometric language, we need to consider the algebraic closure of the field K . Notice however that restricting the field of coefficients does not affect the property of being a Gröbner basis, as long as the polynomials are defined over the smaller field. More precisely, if $I = (g_1, \dots, g_s) \subset K[x_0, \dots, x_n]$ and g_1, \dots, g_s have coefficients in a subfield k of K , then: g_1, \dots, g_s are a Gröbner basis of $I \subseteq K[x_0, \dots, x_n]$ if and only if they are a Gröbner basis of $I \cap k[x_0, \dots, x_n]$. In this sense, the property of being a Gröbner basis does not depend on the field of definition. Therefore, while proving that a set \mathcal{G} of polynomials is a Gröbner basis with respect to a given term-order for the ideal I that it generates, we may pass to the algebraic closure without loss of generality. We shall therefore assume without loss of generality that the field K is algebraically closed.

Notation 1.1. Fix a term-order σ . Let $I \subset R$ be an ideal and let \mathcal{G} be a set of polynomials in R . We denote by $\text{in}(I)$ the initial ideal of I with respect to σ , and by $\text{in}(\mathcal{G})$ the set of initial terms of the elements of \mathcal{G} with respect to σ .

For the convenience of the reader, we recall the definition of diagonal and anti-diagonal term-order.

Definition 1.2. Let X be a matrix (resp. a skew-symmetric or a symmetric matrix) of indeterminates. Let σ be a term-order on the set of terms of $K[X]$. The term-order σ is **diagonal** if the leading term with respect to σ of the determinant of a submatrix of X is the product of the indeterminates on its diagonal. It is **anti-diagonal** if the leading

term with respect to σ of the determinant of a submatrix of X is the product of the indeterminates on its anti-diagonal.

Notation 1.3. Let A be a finitely generated, graded R -module. We denote by $H_A(d)$ the **Hilbert function** of A in degree d , i.e., the dimension of A_d as a k -vector space.

In this paper we study large families of ideals generated by minors or pfaffians in a matrix or a ladder, where the size of the minors or pfaffians is allowed to vary in different regions of the matrix or the ladder. We study their initial ideals with respect to a diagonal or anti-diagonal term-order, and we prove that the associated simplicial complexes are vertex decomposable. In particular, the initial ideals in question are Cohen-Macaulay. For the convenience of the reader, we now recall the main definitions.

Definition 1.4. A **simplicial complex** Δ on $n + 1$ vertices, is a collection of subsets of $\{0, \dots, n\}$ such that for any $F \in \Delta$, if $G \subseteq F$, then $G \in \Delta$. An $F \in \Delta$ is called a **face** of Δ . The dimension of a face F is $\dim F = |F| - 1$, and the **dimension** of the complex is

$$\dim \Delta = \max\{\dim F \mid F \in \Delta\}.$$

The complex $\Delta = 2^{\{0, \dots, n\}}$ is called a **simplex**.

The **vertices** of Δ are the subsets of $\{0, \dots, n\}$ of cardinality one. The faces of Δ which are maximal with respect to inclusion are called **facets**. A complex is **pure** if all its facets have dimension equal to the dimension of the complex.

Notation 1.5. To each face $F \in \Delta$ we associate the following two simplicial subcomplexes of Δ : the **link** of F

$$\text{lk}_F(\Delta) = \{G \in \Delta \mid F \cup G \in \Delta, F \cap G = \emptyset\}$$

and the **deletion**

$$\Delta - F = \{G \in \Delta \mid F \cap G = \emptyset\}.$$

If $F = \{k\}$ is a vertex, we denote the link of F and the deletion by $\text{lk}_k(\Delta)$ and $\Delta - k$, respectively.

Definition 1.6. A simplicial complex Δ is **vertex decomposable** if it is a simplex, or it is the empty set, or there exists a vertex k such that $\text{lk}_k(\Delta)$ and $\Delta - k$ are both pure and vertex decomposable, and

$$\dim \Delta = \dim(\Delta - k) = \dim \text{lk}_k(\Delta) + 1.$$

In this article, we show that the simplicial complexes associated to the initial ideals of the family of ideals that we consider are vertex decomposable.

Definition 1.7. The **Stanley-Reisner ideal** associated to a complex Δ on $n + 1$ vertices is the squarefree monomial ideal

$$I_\Delta = (x_{i_1}, \dots, x_{i_s} \mid \{i_1, \dots, i_s\} \notin \Delta) \subset K[x_0, \dots, x_n].$$

Conversely, to every squarefree monomial ideal $I \subseteq K[x_0, \dots, x_n]$ one can associate the unique simplicial complex $\Delta(I)$ on $n + 1$ vertices, such that $I_{\Delta(I)} = I$.

Remark 1.8. (1) A vertex $\{k\} \in \Delta$ is called a **cone point** if for every face $F \in \Delta$, $F \cup \{k\} \in \Delta$. If Δ has a cone point $\{k\}$, then

$$I_\Delta = I_{\Delta-k}K[x_0, \dots, x_n].$$

Moreover, Δ is vertex decomposable if and only if $\Delta - k$ is. Therefore, *we will not distinguish between a complex and a cone over it.*

- (2) Notice that, if Δ is a complex on $n + 1$ vertices, then both $\text{lk}_k(\Delta)$ and $\Delta - k$ are complexes on n (or fewer) vertices. However, since we do not distinguish between a complex and a cone over it, we will regard them as complexes on $n + 1$ vertices.
- (3) On the side of the associated Stanley-Reisner ideals, let I and J be squarefree monomial ideals such that the generators of I involve fewer variables than the generators of J . Then we may associate to I and J simplicial complexes $\Delta(I)$ and $\Delta(J)$ on the same number of variables. This amounts to regarding I and J as ideals in the same polynomial ring.

We now recall some definitions from liaison theory that will be fundamental throughout the paper.

Definition 1.9. Let $J \subset R$ be a homogeneous, saturated ideal. We say that J is **Gorenstein in codimension $\leq c$** if the localization $(R/J)_P$ is a Gorenstein ring for any prime ideal P of R/J of height smaller than or equal to c . We often say that J is G_c . We call **generically Gorenstein**, or G_0 , an ideal J which is Gorenstein in codimension 0.

Definition 1.10. Let $A \subset B \subset R$ be homogeneous ideals such that $\text{ht } A = \text{ht } B - 1$ and R/A is Cohen-Macaulay. Let $f \in R_d$ be a homogeneous element of degree d such that $A : f = A$. The ideal $C := A + fB$ is called a **Basic Double Link** of degree d of B on A . If moreover A is G_0 and B is unmixed, then C is a **Basic Double G-Link** of B on A .

Definition 1.11. Let $I, J \subset R$ be homogeneous, saturated, unmixed ideals, such that $\text{ht}(I) = \text{ht}(J) = c$. We say that J is obtained by an **elementary biliaison** of height ℓ from I if there exists a Cohen-Macaulay ideal N in R of height $c-1$ such that $N \subseteq I \cap J$ and $J/N \cong [I/N](-\ell)$ as R/N -modules. If in addition the ideal N is G_0 , then J is obtained from I via an **elementary G-biliaison**. If $\ell > 0$ we have an **ascending** elementary G-biliaison.

We refer to [27], [21], and [19] for the basic properties of Basic Double Linkage and elementary biliaison. Notice in particular that, if C is a Basic Double Link of B on A , then it is not known in general whether B and C belong to the same G-liaison class. On the other side, if C is a Basic Double G-Link of B on A , then B and C can be G-linked in two steps.

We are now ready to state a sufficient condition for a set of polynomials to be a Gröbner basis (with respect to a given term-order) for the ideal that they generate.

Lemma 1.12. *Let $I, J, N \subset R$ be homogeneous, saturated, unmixed ideals, such that $N \subseteq I \cap J$ and $\text{ht}(I) = \text{ht}(J) = \text{ht}(N) + 1$. Assume that N is Cohen-Macaulay. Let $A, B, C \subset R$ be monomial ideals such that $C \subseteq \text{in}(J)$, $A = \text{in}(N)$ and $B = \text{in}(I)$ with respect to some*

term-order σ . Assume that A is Cohen-Macaulay and that $\text{ht}(B) = \text{ht}(A) + 1$. Suppose that J is obtained from I via an elementary biliaison of height ℓ on N , and that C is a Basic Double Link of degree ℓ of B on A . Then $C = \text{in}(J)$.

Proof. Since $C \subseteq \text{in}(J)$, it suffices to show that $H_C(d) = H_J(d)$ for all $d \in \mathbb{Z}$. This is indeed the case, since

$$H_C(d) = H_B(d - \ell) + H_A(d) - H_A(d - \ell) = H_I(d - \ell) + H_N(d) - H_N(d - \ell) = H_J(d).$$

□

Remarks 1.13. (1) Notice that, if J is obtained from I via an elementary biliaison on N , we do not know in general whether they belong to the same G-liaison class. However, if in addition N is generically Gorenstein, then J is obtained from I via an elementary G-biliaison on N . In particular, it can be obtained from I via two Gorenstein links on N .

- (2) If in addition A is generically Gorenstein, then C is a Basic Double G-Link of B on A . In particular, it can be obtained from B via two Gorenstein links on A .
- (3) The concepts of Basic Double Linkage and biliaison are interchangeable in the statement of Lemma 1.12. More precisely, Basic Double Linkage is a special case of biliaison. Moreover, it can be shown that if J is obtained from I via an elementary biliaison of height ℓ on N , then there exist an ideal H and a $d \in \mathbb{Z}$ s.t. H is a Basic Double Link of degree $d + \ell$ of I on N and also a Basic Double Link of degree d of J on N . Then it is easy to verify that the lemma holds under the weaker assumption that C is obtained from B via an elementary biliaison of height ℓ on A .

In the next section, we use the lemma to prove that the maximal minors of a matrix of indeterminates are a Gröbner basis of the ideal that they generate with respect to any diagonal term-order. Although the result is well-known, we wish to illustrate our method by showing how it applies to this simple example. In Sections 3, 4, and 5, we apply the lemma to ideals generated by: pfaffians of mixed size in a ladder of a skew-symmetric matrix of indeterminates, minors of mixed size in a ladder of a symmetric matrix of indeterminates, and minors of mixed size in a one-sided ladder of a matrix of indeterminates. We prove that the natural generators of these ideals are a Gröbner basis with respect to any diagonal (in the case of minors of a symmetric matrix) or anti-diagonal (in the case of pfaffians or minors in a generic matrix) term-order. We also prove that their initial ideals are squarefree and that they can be obtained from an ideal generated by indeterminates via a sequence of Basic Double G-links of degree 1, which only involve squarefree monomial ideals. In particular, they are glicci (i.e., they can be obtained from a complete intersections via a sequence of G-links), hence they are Cohen-Macaulay. Moreover, we prove that the simplicial complexes associated to their initial ideals are vertex decomposable.

Notice that, if we knew a priori that the simplicial complexes associated to the initial ideals are vertex decomposable, then we could deduce that the corresponding squarefree monomial ideals are glicci by the following result of Nagel and Römer. However, we cannot directly apply their result in our situation, since we need to first produce the

Basic Double G-links on the squarefree monomial ideals, in order to deduce that the associated simplicial complexes are vertex decomposable.

Theorem 1.14 ([28], Theorem 3.3). *Let Δ be a simplicial complex on $n + 1$ vertices and let $I_\Delta \subset K[x_0, \dots, x_n]$ be the Stanley-Reisner ideal of Δ . Assume that Δ is (weakly) vertex decomposable. Then I_Δ can be obtained from an ideal generated by indeterminates via a sequence of Basic Double G-links of degree 1, which only involve squarefree monomial ideals.*

Notice that, although the statement above is slightly stronger than Theorem 3.3 in [28], the result above follows from the proof in [28].

2. A SIMPLE EXAMPLE: IDEALS OF MAXIMAL MINORS

This section is meant to illustrate the idea and the method of our proof on a simple example. We prove that the maximal minors of a matrix of indeterminates are a Gröbner basis of the ideal that they generate with respect to any diagonal term-order. Notice that for the case of minors in a matrix, diagonal term-orders are the same as anti-diagonal ones, up to transposing the matrix.

Theorem 2.1. *Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$. Let $K[X] = K[x_{ij} \mid 1 \leq i, j \leq n]$ be the polynomial ring associated to X . Let $\mathcal{G}_m(X)$ be the set of maximal minors of X and let $I_m(X) \subset K[X]$ be the ideal generated by $\mathcal{G}_m(X)$. Let σ be a diagonal term-order and let $\text{in}(I_m(X))$ be the initial ideal of $I_m(X)$ with respect to σ . Then $\mathcal{G}_m(X)$ is a reduced Gröbner basis of $I_m(X)$ with respect to σ and $\text{in}(I_m(X))$ is a squarefree, Cohen-Macaulay ideal. Moreover, the simplicial complex Δ_X associated to $\text{in}(I_m(X))$ is vertex decomposable.*

Proof. We proceed by induction on $mn = |X|$. If $|X| = 1$, then the ideal $I_1(X)$ is generated by one indeterminate. Hence $\mathcal{G}_1(X)$ is a reduced Gröbner basis of $I_1(X)$ with respect to any term ordering and $I_1(X) = \text{in}(I_1(X))$ is generated by indeterminates. The associated simplicial complex Δ_X is the empty set, hence it is vertex decomposable.

By induction hypothesis, in order to prove the thesis for a matrix with m rows and n columns, we may assume that it holds for any matrix with fewer than mn entries. If $m = 1$, then $\mathcal{G}_1(X)$ consists of indeterminates, hence it is a reduced Gröbner basis of $I_1(X)$ with respect to any term ordering. Moreover, $I_1(X) = \text{in}(I_1(X))$ is generated by indeterminates. The associated simplicial complex Δ_X is the empty set, hence it is vertex decomposable.

If $m \geq 2$, let $C \subseteq \text{in}(I_m(X))$ be the ideal generated by the initial terms of $\mathcal{G}_m(X)$. We claim that $C = \text{in}(I_m(X))$. In fact, let Z be the $m \times (n - 1)$ matrix obtained from X by deleting the last column, and let Y be the $(m - 1) \times (n - 1)$ matrix obtained from Z by deleting the last row. Let $A = \text{in}(\mathcal{G}_m(Z))$ be the ideal generated by the initial terms of the elements of $\mathcal{G}_m(Z)$, and let $B = \text{in}(\mathcal{G}_{m-1}(Y))$ be the ideal generated by the initial terms of the elements of $\mathcal{G}_{m-1}(Y)$. By the induction hypothesis, $\mathcal{G}_m(Z)$ is a Gröbner basis

for $I_m(Z)$ and $\mathcal{G}_{m-1}(Y)$ is a Gröbner basis for $I_{m-1}(Y)$. In other words, $A = \text{in}(I_m(Z))$ and $B = \text{in}(I_{m-1}(Y))$. Notice that

$$\text{in}(\mathcal{G}_m(X)) = \text{in}(\mathcal{G}_m(Z)) \cup x_{mn} \text{in}(\mathcal{G}_{m-1}(Y))$$

where $x_{mn}\mathcal{G}$ denotes the set of products $x_{mn}g$ for $g \in \mathcal{G}$. Since x_{mn} does not appear in $\text{in}(\mathcal{G}_m(Z))$, we have $A : x_{mn} = A$. Therefore,

$$A + x_{mn}B = C \subseteq \text{in}(I_m(X))$$

and C is a Basic Double G-Link of degree 1 of B on A . A and B are squarefree and glicci by induction hypothesis, therefore C is squarefree and glicci. It follows from [21, Theorem 3.6] that $I_m(X)$ is obtained from $I_{m-1}(Y)$ via an elementary G-biliaison of height 1 on $I_m(Z)$. By Lemma 1.12 the maximal minors of X are a Gröbner basis of $I_m(X)$ with respect to σ , and $C = \text{in}(I_m(X))$.

Finally, let $\Delta_Z, \Delta_Y, \Delta_X$ be the simplicial complexes associated to A, B, C , respectively. Since $A + x_{mn}B = C$,

$$\Delta_Z = \Delta_X - mn \quad \text{and} \quad \Delta_Y = \text{lk}_{mn}(\Delta_X).$$

Since Δ_Y and Δ_Z are vertex decomposable by induction hypothesis, so is Δ_X . \square

Remark 2.2. Theorem 2.1 gives in particular a new proof of the fact that the maximal minors of a generic matrix are a Gröbner basis for the ideal that they generate, with respect to a diagonal or anti-diagonal term-order. This is a classical result. While previous proofs have a combinatorial flavor, our proof is completely algebraic, and independent of all the previous Gröbner basis results.

3. PFAFFIAN IDEALS OF LADDERS

In this section, we study Gröbner bases with respect to an anti-diagonal term-order of ideals generated by pfaffians. We always consider pfaffians in a skew-symmetric matrix whose entries are distinct indeterminates.

Pfaffians of size $2t$ in a skew-symmetric matrix are known to be a Gröbner basis for the ideal that they generate, as shown by Herzog and Trung in [20] and independently by Kurano in [26]. In [9], De Negri generalized this result to pfaffians of size $2t$ in a symmetric ladder. In this section, we extend these results to pfaffians of mixed size in a symmetric ladder. In other words, we consider ideals generated by pfaffians, whose size is allowed to vary in different regions of the ladder (see Definition 3.2). In Theorem 3.7 we prove that the pfaffians are a reduced Gröbner basis with respect to any anti-diagonal term-order for the ideal that they generate, and that the corresponding initial ideal is Cohen-Macaulay and squarefree. Moreover, the associated simplicial complex is vertex decomposable. The proof that we give is not a generalization of the earlier ones. Instead, we use our liaison-theoretic approach and the linkage results of [10].

In the recent paper [11], De Negri and Sbarra consider a different family of ideals generated by pfaffians of mixed size in a skew-symmetric matrix, namely cogenerated ideals. They are able to show that the pfaffians are almost never a Gröbner basis of the ideal that they generate with respect to an anti-diagonal term-order. The family of ideals

that they study and the family that we consider in this article have a small overlap, which consists of ideals of pfaffians of size $2t$ in a symmetric ladder, and of ideals generated by $2t$ -pfaffians in the first m rows and columns of the matrix and $(2t+2)$ -pfaffians in the whole matrix. For the ideals in the overlap, the pfaffians are a Gröbner basis for the ideal that they generate. This follows from Theorem 2.8 of [11], as well as from our Theorem 3.7. The results in [11] and those in this article are obtained independently and by following a completely different approach. Nevertheless, we feel that they complement each other nicely, giving a more complete picture of the behavior of Gröbner bases of pfaffian ideals and of their intrinsic complexity.

Pfaffian ideals of ladders were introduced and studied by De Negri and the first author in [10]. From the point of view of liaison theory, this is a very natural family to consider. In this section, we prove that pfaffians of mixed size in a ladder of a skew-symmetric matrix are a Gröbner basis with respect to any anti-diagonal term-order for the ideal that they generate. We start by introducing the relevant definitions and notation.

Let $X = (x_{ij})$ be an $n \times n$ skew-symmetric matrix of indeterminates. In other words, the entries x_{ij} with $i < j$ are indeterminates, $x_{ij} = -x_{ji}$ for $i > j$, and $x_{ii} = 0$ for all $i = 1, \dots, n$. Let $K[X] = K[x_{ij} \mid 1 \leq i < j \leq n]$ be the polynomial ring associated to X .

Definition 3.1. A **symmetric ladder** \mathcal{L} of X is a subset of the set $\mathcal{X} = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i, j \leq n\}$ with the following properties :

- (1) if $(i, j) \in \mathcal{L}$ then $(j, i) \in \mathcal{L}$,
- (2) if $i < h, j > k$ and $(i, j), (h, k) \in \mathcal{L}$, then also $(i, k), (i, h), (h, j), (j, k) \in \mathcal{L}$.

We do not assume that the ladder \mathcal{L} is connected, nor that X is the smallest skew-symmetric matrix having \mathcal{L} as ladder. It is easy to see that any symmetric ladder can be decomposed as a union of square subladders

$$(1) \quad \mathcal{L} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_s$$

where

$$\mathcal{X}_k = \{(i, j) \mid a_k \leq i, j \leq b_k\},$$

for some integers $1 \leq a_1 \leq \dots \leq a_s \leq n$ and $1 \leq b_1 \leq \dots \leq b_s \leq n$ such that $a_k < b_k$ for all k . We say that \mathcal{L} is the ladder with **upper corners** $(a_1, b_1), \dots, (a_s, b_s)$, and that \mathcal{X}_k is the square subladder of \mathcal{L} with upper outside corner (a_k, b_k) . See Figure 1. We allow two upper corners to have the same first or second coordinate, however we assume that no two upper corners coincide. We assume moreover that all upper corners belong to the border of the ladder, i.e., $(a_k - 1, b_k + 1) \notin \mathcal{L}$. Notice that with these conventions a ladder does not have a unique decomposition of the form (1). In other words, a symmetric ladder does not correspond uniquely to a set of upper corners $(a_1, b_1), \dots, (a_s, b_s)$. However, any symmetric ladder is determined by its upper corners as in (1). Moreover, the upper corners of \mathcal{L} determine the submatrices \mathcal{X}_k . We assume that every symmetric ladder comes with its set of upper corners and the corresponding decomposition as a union of square submatrices as in (1). Notice that the set of upper corners as given in our definition contains all the usual upper outside corners, and may contain some of the usual upper inside corners, as well as other elements of the ladder which are not corners of the ladder in the usual sense.

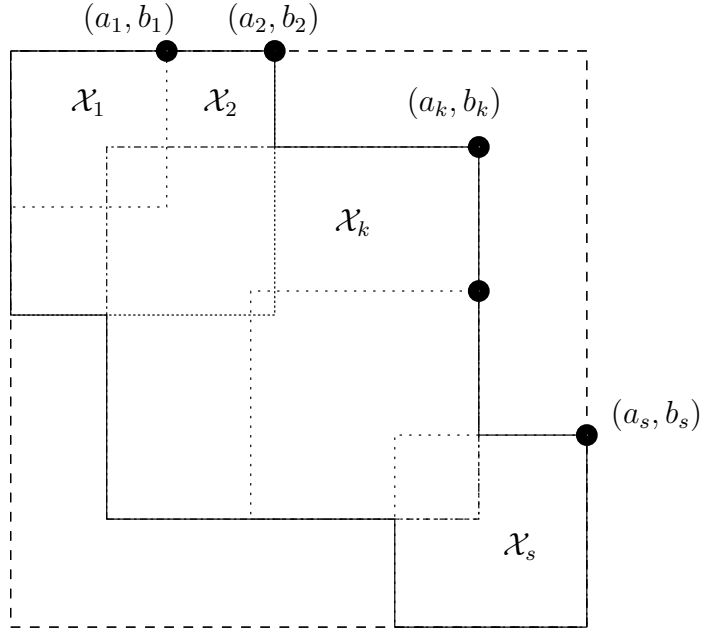


FIGURE 1. An example of a symmetric ladder with its decomposition as a union of skew-symmetric matrices and the corresponding upper corners.

Given a ladder \mathcal{L} we set $L = \{x_{ij} \in X \mid (i, j) \in \mathcal{L}, i < j\}$. If p is a positive integer, we let $I_{2p}(L)$ denote the ideal generated by the set of the $2p$ -pfaffians of X which involve only indeterminates of L . In particular $I_{2p}(X)$ is the ideal of $K[X]$ generated by the $2p$ -pfaffians of X .

Definition 3.2. Let $\mathcal{L} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_s$ be a symmetric ladder. Let $X_k = \{x_{i,j} \mid (i, j) \in \mathcal{X}_k, i < j\}$ for $k = 1, \dots, s$. Fix a vector $t = (t_1, \dots, t_s)$, $t \in \mathbb{Z}_+^s$. The **ladder pfaffian ideal** $I_{2t}(L)$ is by definition the sum of pfaffian ideals $I_{2t_1}(X_1) + \dots + I_{2t_s}(X_s)$. We also refer to these ideals as **pfaffian ideals of ladders**. For ease of notation, we regard all ladder pfaffian ideals as ideals in $K[X]$.

This family of ideals was introduced and studied in [10]. From the point of view of G-biliaison, this appears to be the right family to consider. Notice that it does not coincide with the family of cogenerated pfaffian ideals as defined, e.g., in [8].

Notation 3.3. Denote by $\mathcal{G}_{2t_k}(X_k)$ the set of the $2t_k$ -pfaffians of X which involve only indeterminates of X_k and let

$$\mathcal{G}_{2t}(L) = \mathcal{G}_{2t_1}(X_1) \cup \dots \cup \mathcal{G}_{2t_s}(X_s).$$

The elements of $\mathcal{G}_{2t}(L)$ are a minimal system of generators of $I_{2t}(L)$. We sometimes refer to them as “natural generators”.

Notation 3.4. For a symmetric ladder \mathcal{L} with upper corners $(a_1, b_1), \dots, (a_s, b_s)$ and $t = (t_1, \dots, t_s)$, we denote by $\tilde{\mathcal{L}}$ the symmetric ladder with upper corners $(a_1 + t_1 - 1, b_1 - t_1 + 1), \dots, (a_s + t_s - 1, b_s - t_s + 1)$. See Figure 2.

The ladder $\tilde{\mathcal{L}}$ computes the height of the ideal $I_{2t}(L)$ as follows.

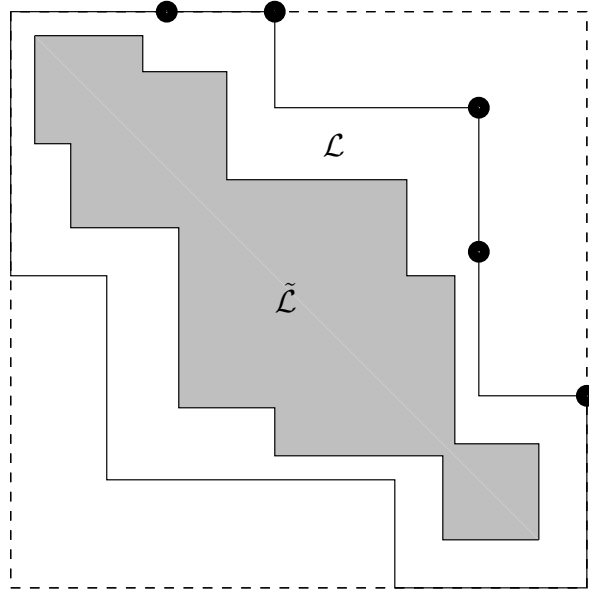


FIGURE 2. An example of a ladder \mathcal{L} with five upper corners and $t = (2, 3, 4, 2, 3)$. The corresponding $\tilde{\mathcal{L}}$ is shaded.

Proposition 3.5 (Proposition 1.10, [10]). *Let \mathcal{L} be the symmetric ladder with upper corners $(a_1, b_1), \dots, (a_s, b_s)$ and $t = (t_1, \dots, t_s)$. Let $\tilde{\mathcal{L}}$ be as in Notation 3.4. Then $\tilde{\mathcal{L}}$ is a symmetric ladder and the height of $I_{2t}(L)$ is equal to the cardinality of $\{(i, j) \in \tilde{\mathcal{L}} \mid i < j\}$.*

The following is the main result of [10]. Its proof consists of an explicit description of the G-biliaison steps, which will be used in the proof of Theorem 3.7.

Theorem 3.6 (Theorem 2.3, [10]). *Any pfaffian ideal of ladders can be obtained from an ideal generated by indeterminates by a finite sequence of ascending elementary G-biliaisons.*

By combining Lemma 1.12 and Theorem 3.6, we prove that the pfaffians are a Gröbner basis of the ideal that they generate with respect to any anti-diagonal term-order.

Theorem 3.7. *Let $X = (x_{ij})$ be an $n \times n$ skew-symmetric matrix of indeterminates. Let $\mathcal{L} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_s$ and $t = (t_1, \dots, t_s)$. Let $I_{2t}(L) \subset K[X]$ be the corresponding ladder pfaffian ideal and let $\mathcal{G}_{2t}(L)$ be the set of pfaffians that generate it. Let σ be any anti-diagonal term-order. Then $\mathcal{G}_{2t}(L)$ is a reduced Gröbner basis of $I_{2t}(L)$ with respect to σ . Moreover, the initial ideal of $I_{2t}(L)$ with respect to σ is squarefree, and the associated simplicial complex is vertex decomposable. In particular, the initial ideal of $I_{2t}(L)$ is Cohen-Macaulay.*

Proof. Let

$$I_{2t}(L) = I_{2t_1}(X_1) + \dots + I_{2t_s}(X_s) \subset K[X]$$

be the pfaffian ideal of the ladder \mathcal{L} with $t = (t_1, \dots, t_s)$ and upper corners $(a_1, b_1), \dots, (a_s, b_s)$. Let $\mathcal{G}_{2t}(L)$ be the set of pfaffians that generate $I_{2t}(L)$. We proceed by induction on $\ell = |\mathcal{L}|$.

If $\ell = |\mathcal{L}| = 1$, then $\mathcal{L} = \tilde{\mathcal{L}}$ and $t = 1$. $\mathcal{G}_{2t}(L)$ consists only of one indeterminate, in particular it is a reduced Gröbner basis with respect to any term-order σ of the ideal that it generates. Since $\text{in}(I_2(L)) = I_2(L)$ is generated by indeterminates, it is squarefree and Cohen-Macaulay, and the associated simplicial complex is the empty set.

We assume that the thesis holds for ideals associated to ladders \mathcal{N} with $|\mathcal{N}| < \ell$ and we prove it for an ideal $I_{2t}(L)$ associated to a ladder \mathcal{L} with $|\mathcal{L}| = \ell$. If $t_1 = \dots = t_s = 1$, then $\mathcal{G}_{2t}(L)$ consists only of indeterminates. In particular, it is a reduced Gröbner basis of the ideal that it generates, with respect to any term-order σ . Moreover, $\text{in}(I_2(L)) = I_2(L)$ is generated by indeterminates, hence it is squarefree and Cohen-Macaulay. The associated simplicial complex is the empty set. Otherwise, let $k \in \{1, \dots, s\}$ such that $t_k = \max\{t_1, \dots, t_s\} \geq 2$. Let \mathcal{L}' be the ladder with upper corners

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k + 1, b_k - 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

and let $t' = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s)$. Let $I_{2t'}(L') \subset K[X]$ be the associated ladder pfaffian ideal. Let $\mathcal{G}_{2t'}(L')$ be the set of pfaffians which minimally generate $I_{2t'}(L')$. Since $|\mathcal{L}'| < \ell$, by induction hypothesis $\mathcal{G}_{2t'}(L')$ is a reduced Gröbner basis of $I_{2t'}(L')$ with respect to any anti-diagonal term-order. Hence

$$\text{in}(I_{2t'}(L')) = (\text{in}(\mathcal{G}_{2t'}(L'))).$$

Let \mathcal{M} be the ladder obtained from \mathcal{L} by removing (a_k, b_k) and (b_k, a_k) . \mathcal{M} has upper corners

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k, b_k - 1), (a_k + 1, b_k), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

and $u = (t_1, \dots, t_{k-1}, t_k, t_k, t_{k+1}, \dots, t_s)$. Let $I_{2u}(M) \subset K[X]$ be the associated ladder pfaffian ideal. Let $\mathcal{G}_{2u}(M)$ be the set of pfaffians which minimally generate $I_{2u}(M)$. Since $|\mathcal{M}| = \ell - 1$, by induction hypothesis $\mathcal{G}_{2u}(M)$ is a reduced Gröbner basis of $I_{2u}(M)$ with respect to any anti-diagonal term-order, and

$$\text{in}(I_{2u}(M)) = (\text{in}(\mathcal{G}_{2u}(M))).$$

It follows from [10], Theorem 2.3 that $I_{2t}(L)$ is obtained from $I_{2t'}(L')$ via an ascending elementary G-biliasion of height 1 on $I_{2u}(M)$. The ideals $\text{in}(I_{2t'}(L'))$ and $\text{in}(I_{2u}(M))$ are Cohen-Macaulay by induction hypothesis. Moreover

$$\text{in}(\mathcal{G}_{2t}(L)) = \text{in}(\mathcal{G}_{2u}(M)) \cup x_{a_k, b_k} \text{in}(\mathcal{G}_{2t'}(L')),$$

where $x_{a,b}\mathcal{G}$ denotes the set of products $x_{a,b}g$ for $g \in \mathcal{G}$. Since x_{a_k, b_k} does not appear in $\text{in}(\mathcal{G}_{2u}(M))$, it does not divide zero modulo the ideal $\text{in}(I_{2u}(M))$. Therefore,

$$(2) \quad I := (\text{in}(\mathcal{G}_{2t}(L))) = \text{in}(I_{2u}(M)) + x_{a_k, b_k} \text{in}(I_{2t'}(L')) \subseteq \text{in}(I_{2t}(L))$$

and I is a Basic Double G-Link of degree 1 of $\text{in}(I_{2t'}(L'))$ on $\text{in}(I_{2u}(M))$. Therefore I is a squarefree Cohen-Macaulay ideal. By Lemma 1.12 $I = \text{in}(I_{2t}(L))$, hence $\mathcal{G}_{2t}(L)$ is a Gröbner basis of $I_{2t}(L)$ with respect to any anti-diagonal term-order.

Let Δ be the simplicial complex associated to $\text{in}(I_{2t}(L))$. By (2) the simplicial complexes associated to $\text{in}(I_{2t'}(L'))$ and $\text{in}(I_{2u}(M))$ are $\text{lk}_{(a_k, b_k)}(\Delta)$ and $\Delta - (a_k, b_k)$, respectively. Δ is vertex decomposable, since $\text{lk}_{(a_k, b_k)}(\Delta)$ and $\Delta - (a_k, b_k)$ are by induction hypothesis. \square

- Remarks 3.8.** (1) From the proof of the theorem it also follows that $I_{2t}(L)$ is obtained from an ideal generated by indeterminates via a sequence of degree 1 Basic Double G-links, which only involve squarefree monomial ideals. Hence in particular it is glicci. Since any vertex decomposable complex is shellable, it also follows that the associated simplicial complex is shellable (see Section 5 of [28] for a summary of the implications among different properties of simplicial complexes, such as vertex decomposability, shellability, Cohen-Macaulayness, etc).
- (2) The proof of Theorem 3.7 given above does not constitute a new proof of the fact that the $2t$ -pfaffians in a matrix or in a symmetric ladder are a Gröbner basis with respect to any anti-diagonal term-order for the ideal that they generate. In fact, our proof is based on Theorem 2.3 in [10], which in turn relies on the fact that pfaffians all of the same size in a ladder of a skew-symmetric matrix generate a prime ideal. Primality of the ideal is classically deduced from the fact that the pfaffians are a Gröbner basis. So we are extending (and not reproving) the results in [20], [26], and [9].

4. SYMMETRIC MIXED LADDER DETERMINANTAL IDEALS

In this section, we study ideals generated by minors contained in a ladder of a generic symmetric matrix. We show that the minors are Gröbner bases for the ideals that they generate, with respect to a diagonal term-order. We also show that the corresponding initial ideal is glicci (hence Cohen-Macaulay) and squarefree, and that the associated simplicial complex is vertex decomposable.

Cogenerated ideals of minors in a symmetric matrix of indeterminates or a symmetric ladder thereof were studied by Conca in [3] and [4]. We refer to [3] and [4] for the definition of cogenerated determinantal ideals in a symmetric matrix. In those articles Conca proved among other things that the natural generators of cogenerated ideals of ladders of a symmetric matrix are a Gröbner bases with respect to any diagonal term-order. In this section, we study the family of symmetric mixed ladder determinantal ideals. This family strictly contains the family of cogenerated ideals. Symmetric mixed ladder determinantal ideals have been introduced and studied by the first author in [18]. This is a very natural family to study, from the point of view of liaison theory. In this paper we extend the result of Conca and prove that the natural generators of symmetric mixed ladder determinantal ideals are a Gröbner bases with respect to any diagonal term-order.

Let $X = (x_{ij})$ be an $n \times n$ symmetric matrix of indeterminates. In other words, the entries x_{ij} with $i \leq j$ are distinct indeterminates, and $x_{ij} = x_{ji}$ for $i > j$. Let $K[X] = K[x_{ij} \mid 1 \leq i \leq j \leq n]$ be the polynomial ring associated to the matrix X . In the sequel, we study ideals generated by the minors contained in a ladder of a generic

symmetric matrix. Throughout the section, we let

$$\mathcal{X} = \{(i, j) \mid 1 \leq i, j \leq n\}.$$

We let \mathcal{L} be a symmetric ladder (see Definition 3.1). We can restrict ourselves to symmetric ladders without loss of generality, since the ideal generated by the minors in a ladder of a symmetric matrix coincides with the ideal generated by the minors in the smallest symmetric ladder containing it. We do not assume that \mathcal{L} is connected, nor that that X is the smallest symmetric matrix having \mathcal{L} as a ladder. Let

$$\mathcal{X}^+ = \{(i, j) \in \mathcal{X} \mid 1 \leq i \leq j \leq n\} \quad \text{and} \quad \mathcal{L}^+ = \mathcal{L} \cap \mathcal{X}^+.$$

Since \mathcal{L} is symmetric, \mathcal{L}^+ determines \mathcal{L} and vice versa. We will abuse terminology and call \mathcal{L}^+ a ladder. Observe that \mathcal{L}^+ can be written as

$$\mathcal{L}^+ = \{(i, j) \in \mathcal{X}^+ \mid i \leq c_l \text{ or } j \leq d_l \text{ for } l = 1, \dots, r \text{ and} \\ i \geq a_l \text{ or } j \geq b_l \text{ for } l = 1, \dots, u\}$$

for some integers $1 \leq a_1 < \dots < a_u \leq n$, $n \geq b_1 > \dots > b_u \geq 1$, $1 \leq c_1 < \dots < c_r \leq n$, and $n \geq d_1 > \dots > d_r \geq 1$, with $a_l \leq b_l$ for $l = 1, \dots, u$ and $c_l \leq d_l$ for $l = 1, \dots, r$.

The points $(a_1, b_2), \dots, (a_{u-1}, b_u)$ are the **upper outside corners** of the ladder, $(a_1, b_1), \dots, (a_u, b_u)$ are the **upper inside corners**, $(c_2, d_1), \dots, (c_r, d_{r-1})$ the **lower outside corners**, and $(c_1, d_1), \dots, (c_r, d_r)$ the **lower inside corners**. If $a_u \neq b_u$, then (a_u, a_u) is an upper outside corner and we set $b_{u+1} = a_u$. Similarly, if $c_r \neq d_r$ then (d_r, d_r) is a lower outside corner, and we set $c_{r+1} = d_r$. A ladder has at least one upper and one lower outside corner. Moreover, $(a_1, b_1) = (c_1, d_1)$ is both an upper and a lower inside corner. See Figure 3. The **lower border** of \mathcal{L}^+ consists of the elements (c, d) of \mathcal{L}^+ such that

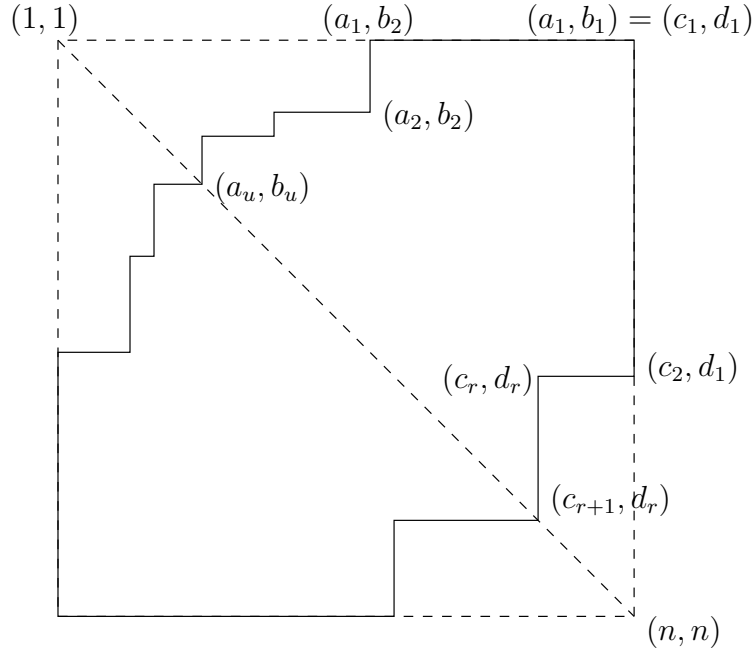


FIGURE 3. An example of ladder with tagged lower and upper corners.

either $c_l \leq c \leq c_{l+1}$ and $d = d_l$, or $c = c_l$ and $d_l \leq d \leq d_{l+1}$ for some l . See Figure 4. All

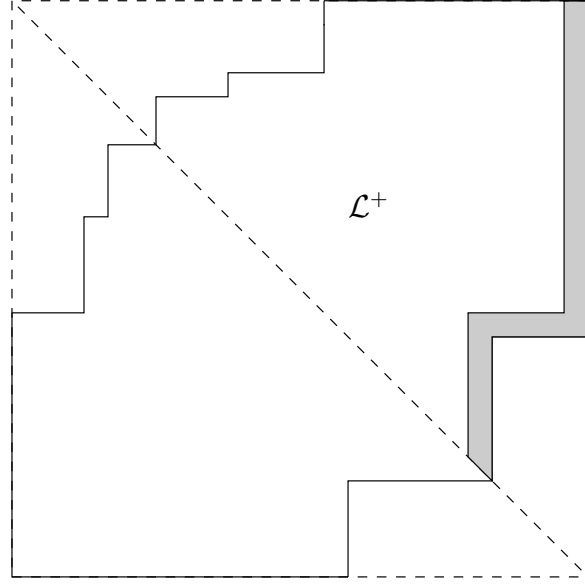


FIGURE 4. The lower border of the same ladder.

the corners belong to \mathcal{L}^+ . In fact, the ladder \mathcal{L}^+ corresponds to its set of lower and upper outside (or equivalently lower and upper inside) corners. The lower corners of a ladder belong to its lower border.

Given a ladder \mathcal{L} we set $L = \{x_{ij} \in X \mid (i, j) \in \mathcal{L}^+\}$. For t a positive integer we let $I_t(L)$ denote the ideal generated by the set of the t -minors of X which involve only indeterminates of L . In particular $I_t(X)$ is the ideal of $K[X]$ generated by the minors of X of size $t \times t$.

Notation 4.1. Let \mathcal{L}^+ be a ladder. For $(v, w) \in \mathcal{L}^+$ let

$$\mathcal{L}_{(v,w)}^+ = \{(i, j) \in \mathcal{L}^+ \mid i \leq v, j \leq w\}, \quad L_{(v,w)} = \{x_{ij} \in X \mid (i, j) \in \mathcal{L}_{(v,w)}^+\}.$$

Notice that $\mathcal{L}_{(v,w)}^+$ is a ladder and

$$\mathcal{L}^+ = \bigcup_{(v,w) \in \mathcal{U}} \mathcal{L}_{(v,w)}^+$$

where \mathcal{U} denotes the set of lower outside corners of \mathcal{L}^+ .

Definition 4.2. Let $\{(v_1, w_1), \dots, (v_s, w_s)\}$ be a subset of the lower border of \mathcal{L}^+ which contains all the lower outside corners. We order them so that $1 \leq v_1 \leq \dots \leq v_s \leq n$ and $n \geq w_1 \geq \dots \geq w_s \geq 1$. Let $t = (t_1, \dots, t_s) \in \mathbb{Z}_+^s$. Denote $\mathcal{L}_{(v_k, w_k)}^+$ by \mathcal{L}_k^+ , and $L_{(v_k, w_k)}$ by L_k . The ideal

$$I_t(L) = I_{t_1}(L_1) + \dots + I_{t_s}(L_s) \subset K[X]$$

is a **symmetric mixed ladder determinantal ideal**. Denote $I_{(t, \dots, t)}(L)$ by $I_t(L)$. We call $(v_1, w_1), \dots, (v_s, w_s)$ **distinguished points** of \mathcal{L}^+ .

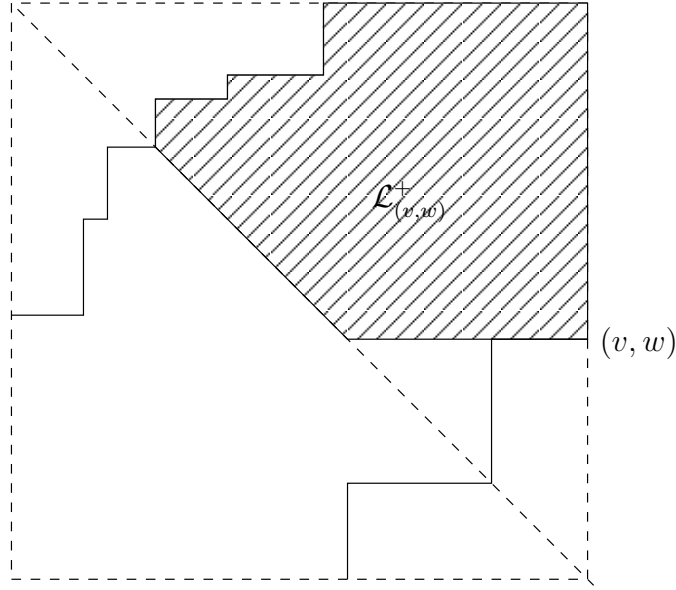


FIGURE 5. The ladder \mathcal{L}^+ with a shaded subladder $\mathcal{L}^+_{(v,w)}$.

If $t = (t, \dots, t)$, then $I_t(L)$ is the ideal generated by the t -minors of X that involve only indeterminates from L . These ideals have been classically studied (see, e.g., [3], [4], [5]). It is not hard to show (see [18], Examples 1.5) that the family of symmetric mixed ladder determinantal ideals contains the family of cogenerated ideals in a ladder of a symmetric matrix, as defined in [4].

Notation 4.3. Denote by $\mathcal{G}_{t_k}(L_k)$ the set of the t_k -minors of X which involve only indeterminates of L_k and let

$$\mathcal{G}_t(L) = \mathcal{G}_{t_1}(L_1) \cup \dots \cup \mathcal{G}_{t_s}(L_s).$$

The elements of $\mathcal{G}_t(L)$ are a minimal system of generators of $I_t(L)$. We sometimes refer to them as “natural generators”.

Notation 4.4. Let \mathcal{L} be a ladder with distinguished points $(v_1, w_1), \dots, (v_s, w_s)$. We denote by

$$\begin{aligned} \tilde{\mathcal{L}}^+ = \{ (i, j) \in \mathcal{L}^+ \mid i \leq v_{k-1} - t_{k-1} + 1 \text{ or } j \leq w_k - t_k + 1 \text{ for } k = 2, \dots, s, \\ j \leq w_1 - t_1 + 1, i \leq v_s - t_s + 1 \} \end{aligned}$$

and by

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^+ \cup \{ (j, i) \mid (i, j) \in \tilde{\mathcal{L}}^+ \}.$$

See Figure 6.

The ladder $\tilde{\mathcal{L}}$ computes the height of $I_t(L)$ as follows.

Proposition 4.5 ([18], Proposition 1.8). *Let \mathcal{L} be a ladder with distinguished points $(v_1, w_1), \dots, (v_s, w_s)$ and let $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^+$ be as above. Then $\tilde{\mathcal{L}}$ is a symmetric ladder and*

$$\text{ht } I_t(L) = |\tilde{\mathcal{L}}^+|.$$

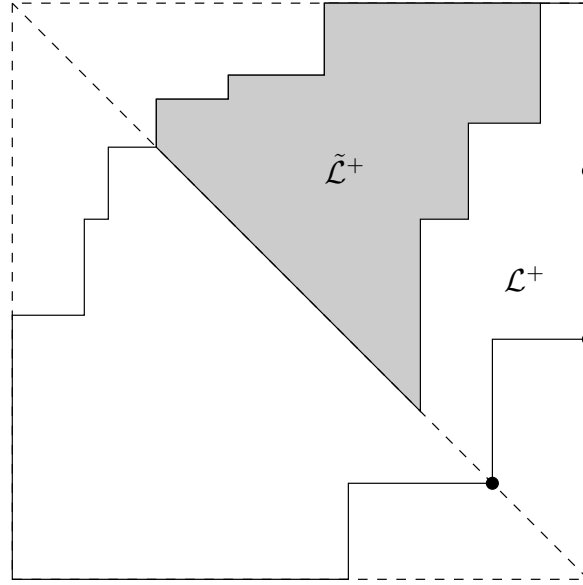


FIGURE 6. An example of \mathcal{L}^+ with three distinguished points and $t = (3, 6, 4)$. The corresponding $\tilde{\mathcal{L}}^+$ is shaded.

The result about Gröbner bases will follow by combining the next theorem with Lemma 1.12.

Theorem 4.6 ([18], Theorem 2.4). *Any symmetric mixed ladder determinantal ideal can be obtained from an ideal generated by indeterminates by a finite sequence of ascending elementary G -biliaisons.*

The following is the main result of this section. We prove that the natural generators of symmetric mixed ladder determinantal ideals are a Gröbner basis with respect to any diagonal term-order, and that the simplicial complexes associated to their initial ideals are vertex decomposable. In particular, the initial ideals are Cohen-Macaulay.

Theorem 4.7. *Let $X = (x_{ij})$ be an $n \times n$ symmetric matrix of indeterminates. Let $\mathcal{L}^+ = \mathcal{L}_1^+ \cup \dots \cup \mathcal{L}_s^+$ and $t = (t_1, \dots, t_s)$. Let $I_t(L) \subset K[X]$ be the corresponding symmetric mixed ladder determinantal ideal and let $\mathcal{G}_t(L)$ be the set of minors that generate it. Let σ be any diagonal term-order. Then $\mathcal{G}_t(L)$ is a reduced Gröbner basis of $I_t(L)$ with respect to σ . Moreover, the initial ideal of $I_t(L)$ with respect to σ is squarefree and Cohen-Macaulay, and the associated simplicial complex is vertex decomposable.*

Proof. Let

$$I_t(L) = I_{t_1}(L_1) + \dots + I_{t_s}(L_s) \subset K[X]$$

be the symmetric mixed ladder determinantal ideal with ladder \mathcal{L} , $t = (t_1, \dots, t_s)$ and distinguished points $(v_1, w_1), \dots, (v_s, w_s)$. Let $\mathcal{G}_t(L)$ be the set of natural generators of $I_t(L)$. We proceed by induction on $\ell = |\mathcal{L}^+|$.

If $\ell = 1$, then $\mathcal{G}_t(L)$ consists of one indeterminate, in particular it is a Gröbner basis of the ideal that it generates with respect to any term-order σ . Moreover, $\text{in}(I_1(L)) =$

$I_1(L)$ is generated by indeterminates, and the simplicial complex associated to it is the empty set.

We now assume that the thesis holds for ladders \mathcal{N} with $|\mathcal{N}^+| < \ell$ and prove it for a ladder \mathcal{L} with $|\mathcal{L}^+| = \ell$. If $t_1 = \dots = t_s = 1$, then $\mathcal{G}_t(L)$ consists only of indeterminates. In particular it is a Gröbner basis of the ideal that it generates with respect to any term-order σ . Moreover, $\text{in}(I_1(L)) = I_1(L)$ is generated by indeterminates, and the simplicial complex associated to it is the empty set. Otherwise, let $k \in \{1, \dots, s\}$ such that $t_k = \max\{t_1, \dots, t_s\} \geq 2$. Let \mathcal{L}' be the ladder with distinguished points

$$(v_1, w_1), \dots, (v_{k-1}, w_{k-1}), (v_k + 1, w_k - 1), (v_{k+1}, w_{k+1}), \dots, (v_s, w_s)$$

and let $t' = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s)$. Let $I_{t'}(L') \subset K[X]$ be the associated symmetric mixed ladder determinantal ideal. Let $\mathcal{G}_{t'}(L')$ be the set of minors which minimally generate $I_{t'}(L')$. Since $|\mathcal{L}'^+| < \ell$, by induction hypothesis $\mathcal{G}_{t'}(L')$ is a reduced Gröbner basis of $I_{t'}(L')$ with respect to σ . Hence

$$\text{in}(I_{t'}(L')) = (\text{in}(\mathcal{G}_{t'}(L'))).$$

Let \mathcal{M} be the ladder obtained from \mathcal{L} by removing (v_k, w_k) and (w_k, v_k) . Let

$$(v_1, w_1), \dots, (v_{k-1}, w_{k-1}), (v_k, w_k - 1), (v_k + 1, w_k), (v_{k+1}, w_{k+1}), \dots, (v_s, w_s)$$

be the distinguished points of \mathcal{M} and let $u = (t_1, \dots, t_{k-1}, t_k, t_k, t_{k+1}, \dots, t_s)$. Let $I_u(M) \subset K[X]$ be the associated symmetric mixed ladder determinantal ideal. Let $\mathcal{G}_u(M)$ be the set of minors which minimally generate $I_u(M)$. Since $|\mathcal{M}^+| = \ell - 1 < \ell$, by induction hypothesis $\mathcal{G}_u(M)$ is a reduced Gröbner basis of $I_u(M)$ with respect to any diagonal term-order. Hence

$$\text{in}(I_u(M)) = (\text{in}(\mathcal{G}_u(M))).$$

It follows from [18], Theorem 2.4 that $I_t(L)$ is obtained from $I_{t'}(L')$ via an ascending elementary G-biliaison of height 1 on $I_u(M)$. The ideals $\text{in}(I_{t'}(L'))$ and $\text{in}(I_u(M))$ are squarefree and Cohen-Macaulay by induction hypothesis. Moreover

$$\text{in}(\mathcal{G}_t(L)) = \text{in}(\mathcal{G}_u(M)) \cup x_{v_k, w_k} \text{in}(\mathcal{G}_{t'}(L')),$$

where $x_{u,v}\mathcal{G}$ denotes the set of products $x_{u,v}g$ for $g \in \mathcal{G}$. Since x_{v_k, w_k} does not appear in $\text{in}(\mathcal{G}_u(M))$, it does not divide zero modulo the ideal $\text{in}(I_u(M))$. Therefore,

$$(3) \quad I := (\text{in}(\mathcal{G}_t(L))) = \text{in}(I_u(M)) + x_{v_k, w_k} \text{in}(I_{t'}(L')) \subseteq \text{in}(I_t(L))$$

and I is a Basic Double G-Link of degree 1 of $\text{in}(I_{t'}(L'))$ on $\text{in}(I_u(M))$. Hence I is a squarefree Cohen-Macaulay ideal. By Lemma 1.12 $I = \text{in}(I_t(L))$, hence $\mathcal{G}_t(L)$ is a Gröbner basis of $I_t(L)$ with respect to any diagonal term-order.

Let Δ be the simplicial complex associated to $\text{in}(I_t(L))$. By (3) the simplicial complexes associated to $\text{in}(I_{t'}(L'))$ and $\text{in}(I_u(M))$ are $\text{lk}_{(v_k, w_k)}(\Delta)$ and $\Delta - (v_k, w_k)$, respectively. Δ is vertex decomposable, since $\text{lk}_{(v_k, w_k)}(\Delta)$ and $\Delta - (v_k, w_k)$ are by induction hypothesis. \square

Remarks 4.8. (1) From the proof of the previous theorem it also follows that $\text{in}_t(L)$ is obtained from an ideal generated by indeterminates via a sequence of degree 1 Basic Double G-links which only involve squarefree monomial ideals. In particular, it is glicci. Moreover, the associated simplicial complex is shellable.

- (2) The proof of Theorem 4.7 given above does not constitute a new proof of the fact that the t -minors in a symmetric matrix or in a symmetric ladder are a Gröbner basis with respect to any diagonal term-order for the ideal that they generate. In fact, our proof is based on Theorem 2.4 in [18], which in turn relies on the fact that minors all of the same size in a ladder of a symmetric matrix generate a prime ideal. Primality of this ideal is classically deduced from the fact that the minors are a Gröbner basis. So we are extending (and not providing a new proof of) the results in [3].
- (3) Our argument, however, gives a new proof of the fact that the minors generating a cogenerated ideal in a symmetric matrix or in a ladder thereof are a Gröbner basis with respect to a diagonal term-order, knowing that minors all of the same size in a ladder of a symmetric matrix are a Gröbner basis of the ideal that they generate.

5. MIXED LADDER DETERMINANTAL IDEALS

In this section, we prove that minors of mixed size in one-sided ladders are Gröbner bases for the ideals that they generate, with respect to any anti-diagonal term order. Moreover, the associated simplicial complex is vertex decomposable. These results are already known, and were established in different levels of generality in [29], [2], [6], [14], [22], [16], [23] and [24]. The papers [22], [23] and [24] follow a different approach than the others. The family that they treat strictly contains that of one-sided mixed ladder determinantal ideals. The paper [16] follows essentially the same approach as the the first four papers, extending it to the family of two-sided mixed ladder determinantal ideals. The proof we give here is different and independent of all the previous ones: we use the result that we established in Section 1 and the liaison results which were established in [16].

In [16] the first author approached the study of ladder determinantal ideals from the opposite point of view: she first proved that the minors were Gröbner bases of the ideals that they generated. From the Gröbner basis result, she deduced that the ideals are prime and Cohen-Macaulay, and computed their height. Finally, she proved the liaison result. Here we wish to take the opposite approach: namely, deduce the fact that the minors are a Gröbner basis for the ideal that they generate from the liaison result. In order to do that, we need to show how to obtain the liaison result independently of the computation of a Gröbner basis. We do this in Appendix A following the approach of [10] and [18]. In this section, we deduce the result about Gröbner bases from the G-biliaison result.

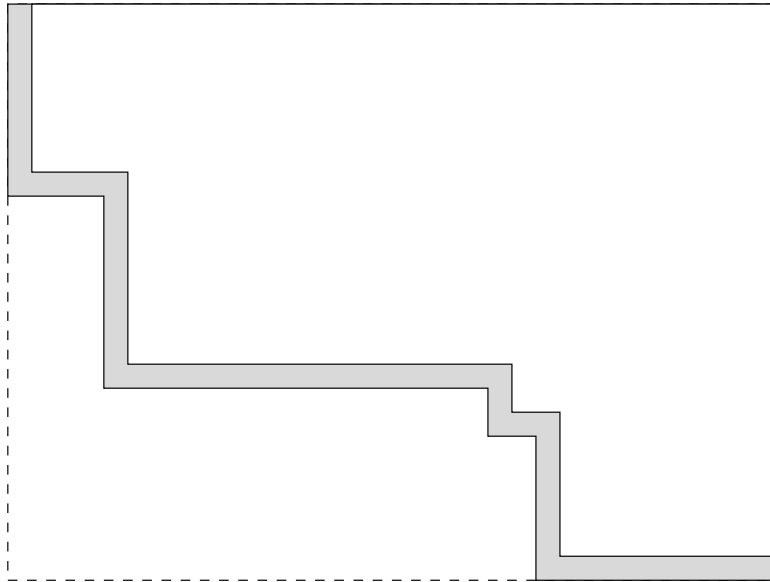
We start by introducing the relevant notation. Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$.

Definition 5.1. A **one-sided ladder** \mathcal{L} of X is a subset of the set $\mathcal{X} = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ with the properties:

- (1) $(1, m) \in \mathcal{L}$,
- (2) if $i < h, j > k$ and $(i, j), (h, k) \in \mathcal{L}$, then $(i, k), (i, h), (h, j), (j, k) \in \mathcal{L}$.

$$\mathcal{L} = \bigcup_{k=1}^u \{(i, j) \in \mathcal{X} \mid i \leq c_k \text{ and } j \geq d_k\}$$

We call $(c_1, d_1), \dots, (c_u, d_u)$ **lower outside corners** and $(c_1, d_2), \dots, (c_{u-1}, d_u)$ **lower inside corners** of the ladder \mathcal{L} . A one-sided ladder has at least one lower outside corner. A one-sided ladder which has exactly one lower outside corner is a matrix. All the corners belong to \mathcal{L} , and the ladder \mathcal{L} corresponds to its set of lower outside (or equivalently, lower inside) corners. The **lower border** of \mathcal{L} consists of the elements (c, d) of \mathcal{L} such that $(c + 1, d - 1) \notin \mathcal{L}$. See Figure 7. Notice that the lower corners of a ladder belong to its lower border.



Given a ladder \mathcal{L} we set $L = \{x_{ij} \in X \mid (i, j) \in \mathcal{L}\}$. We denote by $|\mathcal{L}|$ the cardinality of the ladder. We let $I_t(L)$ denote the ideal generated by the set of the t -minors of X which involve only indeterminates of L . In particular, $I_t(X)$ is the ideal of $K[X]$ generated by the $t \times t$ -minors of X .

$$\mathcal{L}_k = \{(i, j) \in \mathcal{X} \mid i \leq a_k \text{ and } j \geq b_k\} \quad \text{and} \quad L_k = \{x_{i,j} \mid (i, j) \in \mathcal{L}_k\}.$$

Notice that $\mathcal{L}_k \subseteq \mathcal{L}$ and $L_k \subseteq L$. Moreover, $\mathcal{L} = \cup_{k=1}^s \mathcal{L}_k$. The ideal

$$I_t(L) = I_{t_1}(L_1) + \dots + I_{t_s}(L_s)$$

is a **mixed ladder determinantal ideal**. We denote $I_{(t_1, \dots, t_s)}(L)$ by $I_t(L)$. We call $(a_1, b_1), \dots, (a_s, b_s)$ **distinguished points** of \mathcal{L} . Notice that a ladder is uniquely determined by the set of its distinguished points, but it does not determine them.

Notation 5.3. Denote by $\mathcal{G}_{t_k}(L_k)$ the set of the t_k -minors of X which involve only indeterminates of L_k and let

$$\mathcal{G}_t(L) = \mathcal{G}_{t_1}(L_1) \cup \dots \cup \mathcal{G}_{t_s}(L_s).$$

The elements of $\mathcal{G}_t(L)$ are a minimal system of generators of $I_t(L)$. We sometimes refer to them as “natural generators”.

We will need the following result. See the appendix for a self contained proof.

Theorem 5.4 ([16], Theorem 2.1). *Any mixed ladder determinantal ideal can be obtained from an ideal generated by indeterminates by a finite sequence of ascending elementary G -biliaisons.*

We now prove that the natural generators of a mixed ladder determinantal ideal are a Gröbner basis with respect to any anti-diagonal term-order.

Theorem 5.5. *Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$, and let \mathcal{L} be a one-sided ladder of X . Let $t \in \mathbb{N}^s$ and let $(a_1, b_1), \dots, (a_s, b_s)$ be the distinguished points of the ladder. Let $I_t(L)$ be the corresponding ladder determinantal ideal. Denote by $\mathcal{G}_t(L)$ be the set of minors that generate $I_t(L)$. Then $\mathcal{G}_t(L)$ is a reduced Gröbner basis of $I_t(L)$ with respect to any anti-diagonal term ordering. Moreover, the initial ideal $\text{in}(I_t(L))$ is squarefree Cohen-Macaulay, and the associated simplicial complex is vertex decomposable.*

Proof. We proceed by induction on $\ell = |\mathcal{L}|$. If $\ell = 1$, then $\mathcal{G}_1(L)$ consists of one indeterminate. Hence $\mathcal{G}_1(L)$ is a reduced Gröbner basis for $I_1(L)$ with respect to any term-order. Moreover, $I_1(L) = \text{in}(I_1(L))$ is generated by indeterminates, hence the associated simplicial complex is the empty set.

We now assume by induction that the statement holds for ladders \mathcal{H} with $|\mathcal{H}| < \ell$, and we prove the statement for a ladder \mathcal{L} with $|\mathcal{L}| = \ell$. If $t = (1, \dots, 1)$, then $\mathcal{G}_1(L)$ consists of indeterminates. Hence $\mathcal{G}_1(L)$ is a reduced Gröbner basis for $I_1(L)$ with respect to any term-order. Moreover, $I_1(L) = \text{in}(I_1(L))$ is generated by indeterminates, hence the associated simplicial complex is the empty set. Otherwise, let $C \subseteq \text{in}(I_t(L))$ be the ideal generated by the initial terms of $\mathcal{G}_t(L)$. It suffices to show that $C = \text{in}(I_t(L))$ and that C is Cohen-Macaulay. Let $(a_1, b_1), \dots, (a_s, b_s)$ be the distinguished points of the ladder and choose $k \in \{1, \dots, s\}$ so that $t_k = \max\{t_1, \dots, t_s\} \geq 2$. Let $\mathcal{M} = \mathcal{L} \setminus \{(a_k, b_k)\}$, and let

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k - 1, b_k), (a_k, b_k + 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

be the distinguished points of \mathcal{M} . Let $p = (t_1, \dots, t_{k-1}, t_k, t_k, t_{k+1}, \dots, t_s) \in \mathbb{N}^{s+1}$. Let \mathcal{N} be the ladder with distinguished points

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k - 1, b_k + 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

and let $q = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s) \in \mathbb{N}^s$. As shown in the proof of Theorem 5.4, $I_t(L)$ is obtained from $I_q(N)$ via an elementary G-biliaison of height 1 on $I_p(M)$.

Let $A = \text{in}(\mathcal{G}_p(M))$ and let $B = \text{in}(\mathcal{G}_q(N))$. The induction hypothesis applies to both $I_p(M)$ and $I_q(N)$, hence $\mathcal{G}_p(M)$ is a Gröbner basis for $I_p(M)$, $\mathcal{G}_q(N)$ is a Gröbner basis for $I_q(N)$ and $A = \text{in}(I_p(M))$, $B = \text{in}(I_q(N))$ are Cohen-Macaulay ideals. Notice that

$$\text{in}(\mathcal{G}_t(L)) = \text{in}(\mathcal{G}_p(M)) \cup x_{a_k, b_k} \text{in}(\mathcal{G}_q(N))$$

where $x_{a_k, b_k} \mathcal{G}$ denotes the set of products $x_{a_k, b_k} g$ for $g \in \mathcal{G}$. Since x_{a_k, b_k} does not appear in $\text{in}(\mathcal{G}_p(M))$, it does not divide zero modulo the ideal $A = \text{in}(I_p(M))$. Therefore,

$$(4) \quad \text{in}(I_p(M)) + x_{a_k, b_k} \text{in}(I_q(N)) = C \subseteq \text{in}(I_t(L))$$

and C is a Basic Double G-Link of degree 1 of $\text{in}(I_q(N))$ on $\text{in}(I_p(M))$. C is Cohen-Macaulay and squarefree, since A and B are. By Lemma 1.12 we conclude that $C \subseteq \text{in}(I_t(L))$ and that $\mathcal{G}_t(L)$ is a Gröbner basis of $I_t(L)$ with respect to any anti-diagonal term-order.

Let Δ be the simplicial complex associated to $\text{in}(I_t(L))$. By (4) the simplicial complexes associated to $\text{in}(I_q(N)$ and $\text{in}(I_p(M))$ are $\text{lk}_{(a_k, b_k)}(\Delta)$ and $\Delta - (a_k, b_k)$, respectively. Δ is vertex decomposable, since $\text{lk}_{(a_k, b_k)}(\Delta)$ and $\Delta - (a_k, b_k)$ are by induction hypothesis. \square

Remarks 5.6. (1) From the proof of the theorem it also follows that the ideal $\text{in}(I_t(L))$ is obtained from an ideal generated by indeterminates via a sequence of degree 1 Basic Double G-links, which only involve squarefree monomial ideals. Hence in particular it is glicci. Moreover, the associated simplicial complex is shellable.

(2) Notice that, in contrast to Theorem 3.7 and Theorem 4.7, Theorem 5.5 does constitute a new proof of the fact that t -minors in a generic matrix or in a one-sided ladder are a Gröbner basis with respect to any anti-diagonal term-order for the ideal that they generate. In fact, in Theorem A.5 we give a proof of primality for mixed ladder determinantal ideals which is independent of any previous Gröbner basis results.

By following the same approach as in the previous sections and using the result of Narasimhan from [29], we can prove that the natural generators of mixed ladder determinantal ideals from two-sided ladders are a Gröbner basis for the ideal that they generate with respect to any anti-diagonal term-order (see [16] for the relevant definitions). This is, to our knowledge, the largest family of ideals generated by minors in a ladder for which the minors are a Gröbner basis for the ideal that they generate. Notice, e.g., that cogenerated ladder determinantal ideals all belong to this family. The result was already established by the first author in [16], but a different proof can be given using the techniques discussed in this paper. Notice moreover that we also show that the simplicial complex associated to the initial ideal is vertex decomposable. In particular, it is shellable. Since the proof is completely analogous to the previous ones, we omit it.

Theorem 5.7. *Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$, and let \mathcal{L} be a ladder of X . Let $t \in \mathbb{N}^s$ and let $(a_1, b_1), \dots, (a_s, b_s)$ be the distinguished points of the ladder. Let $I_t(L)$ be the corresponding ladder determinantal*

ideal. Denote by $\mathcal{G}_t(L)$ be the set of minors that generate $I_t(L)$. Then $\mathcal{G}_t(L)$ is a reduced Gröbner basis of $I_t(L)$ with respect to any anti-diagonal term ordering, and the initial ideal $\text{in}(I_t(L))$ is squarefree Cohen-Macaulay. Moreover, the associated simplicial complex is vertex decomposable.

APPENDIX A. G-BILIAISON OF MIXED LADDER DETERMINANTAL IDEALS FROM ONE-SIDED LADDERS

Mixed ladder determinantal ideals were introduced and studied by the first author in [16]. They are a natural family to study, from the point of view of liaison theory. In [16], the first author proved that the minors were Gröbner bases of the ideals that they generated. From the Gröbner basis result, she deduced that the ideals are prime, and Cohen-Macaulay, and computed their height. Finally, she proved the liaison result. In this article we wish to take the opposite approach: namely, deduce the fact that the minors are a Gröbner basis for the ideal that they generate from the liaison result. In order to do that, we need to show how to obtain the liaison result independently of the computation of a Gröbner basis. We do this by showing that the approach of [10] (for ladder ideals of pfaffians) and [18] (for mixed symmetric determinantal ideals) applies also to mixed ladder determinantal ideals. More precisely, we prove that mixed ladder determinantal ideals from one-sided ladders are prime and Cohen-Macaulay. We also give a different proof of the formula for their height which was given in [16]. We do all these without relying on, and independently of, the computation of a Gröbner basis. The liaison results of [16] then follow, in particular we obtain a proof of Theorem 5.4 which does not rely on any Gröbner basis results.

We follow the definitions and notations of Section 5.1. The following easily follow from the definition of a mixed ladder determinantal ideal.

Remarks A.1. [[16], Assumption 3 and Lemma 1.13]

- (1) We may assume without loss of generality that $t_k \leq \min\{a_k, m - b_k + 1\}$ for $1 \leq k \leq s$.
- (2) We may assume that $b_{k-1} - b_k < t_k - t_{k-1} < a_k - a_{k-1}$ for $k \geq 2$.

Notation A.2. Let \mathcal{L} be a one-sided ladder with distinguished points $(a_1, b_1), \dots, (a_s, b_s)$ and $t = (t_1, \dots, t_s)$. We denote by $\tilde{\mathcal{L}}$ the one-sided ladder with lower outside corners

$$(a_1 - t_1 + 1, b_1 + t_1 - 1), \dots, (a_s - t_s + 1, b_s + t_s - 1).$$

$\tilde{\mathcal{L}}$ is a ladder by Remarks A.1 (2). See Figure 8.

The height of ladder determinantal ideals from one-sided ladders was first computed by Gonciulea and Miller in [14], Theorem 4.6.3. In Theorem 1.15 of [16], the first author gave a new formula for the height of ladder determinantal ideals from ladders which are not necessarily one-sided. More precisely, she proved that

$$(5) \quad \text{ht } I_t(L) = |\tilde{\mathcal{L}}|.$$

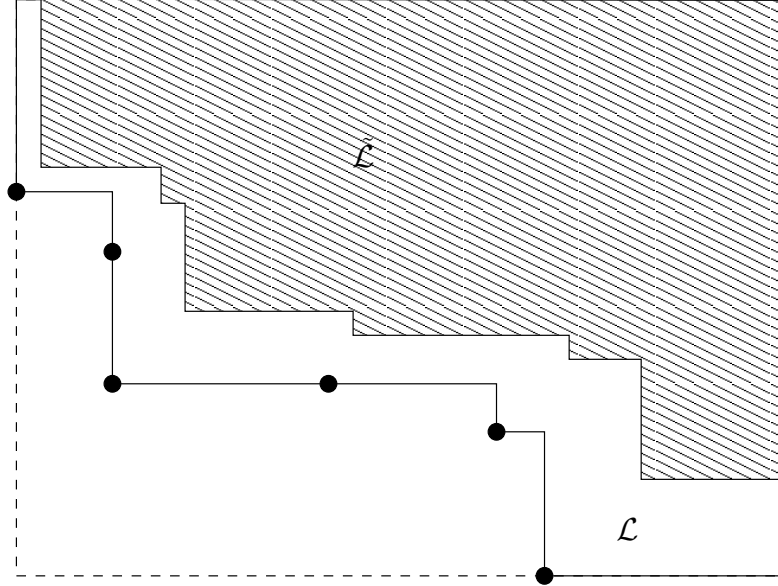


FIGURE 8. A ladder \mathcal{L} with marked distinguished points. The subladder $\tilde{\mathcal{L}}$ is shaded.

Both the proofs in [14] and [16] relied on the computation of a Gröbner basis of the ladder determinantal ideal. In Theorem A.5 we give a different proof of the height formula (5), which is independent of the computation of a Gröbner basis.

Notation A.3. Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$, and let $\mathcal{L} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_s$ be a one-sided ladder of X with distinguished points $(a_1, b_1), \dots, (a_s, b_s)$. Let $(u, v) \in \mathcal{L}$ and assume that $(u, v) \in \mathcal{L}_i$ for $j \leq i \leq k$ only. We let $\hat{\mathcal{L}}$ denote the ladder obtained as follows: remove the entries in row u and column v which do not belong to any \mathcal{L}_i for $i \notin \{j, \dots, k\}$. The remaining entries in row u and column v are $(1, v), \dots, (a_{j-1}, v) \in \mathcal{L}_{j-1}$ and $(u, b_{k+1}), \dots, (u, n) \in \mathcal{L}_{k+1}$. Move the remaining entries in row u just below region \mathcal{L}_k , i.e., between row a_k and row $a_k + 1$. Move the remaining entries in column v just on the left of region \mathcal{L}_j , i.e., between column $b_j - 1$ and column b_j . It is easy to check that $\hat{\mathcal{L}}$ is a ladder. Rename the entries of the ladder as needed, so that (a, b) denotes the entry in position (a, b) . Finally, let

$$(a_1, b_1), \dots, (a_{j-1}, b_{j-1}), (a_j - 1, b_j + 1), \dots, (a_k - 1, b_k + 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

be the distinguished points of $\hat{\mathcal{L}}$. See also Figure 9.

We now prove a technical lemma which will be needed in the proof of primality. The statement is essentially contained in [14], but here we prove it for a larger family. The technique of the proof is a standard one, and can be found, e.g., in [1].

Lemma A.4. Let $X = (x_{ij})$ be an $m \times n$ matrix whose entries are distinct indeterminates, $m \leq n$, and let \mathcal{L} be a one-sided ladder of X . Let $t \in \mathbb{N}^s$ and let

$$(a_1, b_1), \dots, (a_s, b_s)$$

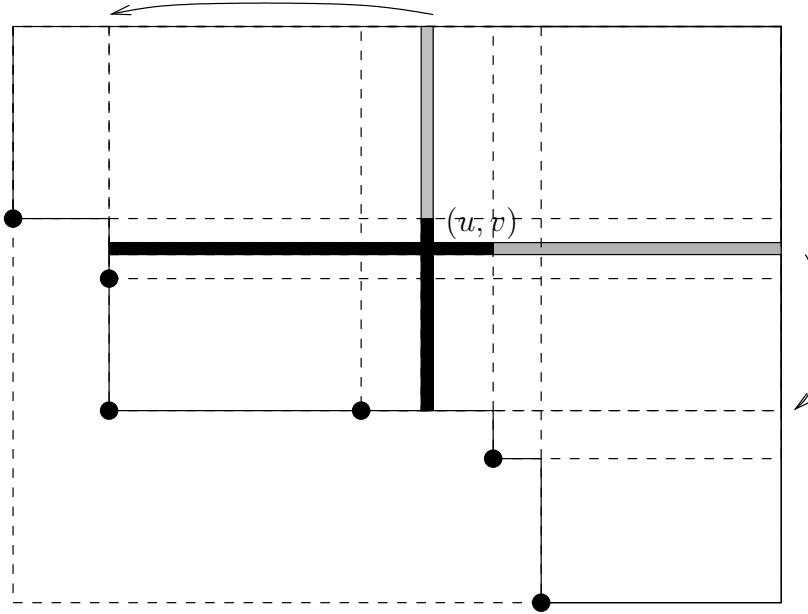


FIGURE 9. An example of the construction of $\hat{\mathcal{L}}$. The entries in black are deleted, while the ones in gray are moved as indicated by the arrows.

be the distinguished points of the ladder. Let $I_t(L)$ be the corresponding ladder determinantal ideal. Let $(u, v) \in \mathcal{L}$ be a point of the ladder. Let $\hat{\mathcal{L}}$ be the ladder obtained as in Notation A.3, with distinguished points

$$(a_1, b_1), \dots, (a_{j-1}, b_{j-1}), (a_j - 1, b_j + 1), \dots, (a_k - 1, b_k + 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s).$$

Assume that $t_i \geq 2$ for $j \leq i \leq k$ and let

$$r = (t_1, \dots, t_{j-1}, t_j - 1, \dots, t_k - 1, t_{k+1}, \dots, t_s).$$

Then there is an isomorphism

$$K[L]/I_t(L)[x_{u,v}^{-1}] \cong K[\hat{L}]/I_r(\hat{L})[x_{a_{j-1}+1,v}, \dots, x_{a_k,v}, x_{u,b_j}, \dots, x_{u,b_{k+1}-1}, x_{u,v}^{-1}].$$

Proof. Under our assumptions, $\hat{\mathcal{L}}$ is a ladder and $I_r(\hat{L})$ is a mixed ladder determinantal ideal. Let $A = K[L][x_{u,v}^{-1}]$ and

$$B = K[\hat{L}][x_{a_{j-1}+1,v}, \dots, x_{a_k,v}, x_{u,b_j}, \dots, x_{u,b_{k+1}-1}, x_{u,v}^{-1}].$$

Define a K -algebra homomorphism

$$\begin{aligned} \varphi : A &\longrightarrow B \\ x_{i,j} &\longmapsto \begin{cases} x_{i,j} + x_{i,v}x_{u,j}x_{u,v}^{-1} & \text{if } i \neq u, j \neq v \text{ and } (i,j) \in \mathcal{L}_j \cup \dots \cup \mathcal{L}_k, \\ x_{i,j} & \text{otherwise.} \end{cases} \end{aligned}$$

The inverse of φ is

$$\begin{aligned} \psi : B &\longrightarrow A \\ x_{i,j} &\longmapsto \begin{cases} x_{i,j} - x_{i,v}x_{u,j}x_{u,v}^{-1} & \text{if } i \neq u, j \neq v \text{ and } (i,j) \in \hat{\mathcal{L}}_j \cup \dots \cup \hat{\mathcal{L}}_k, \\ x_{i,j} & \text{otherwise.} \end{cases} \end{aligned}$$

It is easy to check that φ and ψ are inverse to each other. Since

$$\varphi(I_{t_i}(L_i)A) = I_{t_i-1}(\hat{L}_i)B$$

for $j \leq i \leq k$, we have

$$\varphi(I_t(L)A) = I_r(\hat{L})B \quad \text{hence} \quad A/I_t(L)A \cong B/I_r(\hat{L})B.$$

□

We now prove that mixed ladder determinantal ideals are prime and Cohen-Macaulay. We also give a proof of the height formula (5). For the sake of clarity, we repeat the proof that mixed ladder determinantal ideals are obtained from a linear space by a finite sequence of ascending elementary G-biliaisons. The proof of primality is adapted from the proof of Bruns and Vetter for ideals of maximal minors of a matrix ([1], Theorem 2.10).

Theorem A.5 ([16]; Theorem 1.15, Theorem 1.18, Theorem 1.21, Theorem 2.1). *Let \mathcal{L} be a one-sided ladder of X . Let $(a_1, b_1), \dots, (a_s, b_s)$ be its distinguished points and let $t \in \mathbb{N}^s$. Let $I_t(L) \subset K[X]$ be the corresponding ladder determinantal ideal. Then:*

- (1) $I_t(L)$ is prime and Cohen-Macaulay.
- (2) Let $\tilde{\mathcal{L}}$ be the one-sided ladder of Notation A.2, i.e., the ladder with lower outside corners $(a_1 - t_1 + 1, b_1 + t_1 - 1), \dots, (a_s - t_s + 1, b_s + t_s - 1)$. Then $I_t(L) \subset K[\tilde{\mathcal{L}}]$ has height

$$\text{ht}(I_t(L)) = |\tilde{\mathcal{L}}|.$$

- (3) $I_t(L)$ can be obtained from an ideal generated by indeterminates by a finite sequence of ascending elementary G-biliaisons.

Proof. We prove the statement by induction on $\ell = |\mathcal{L}|$. If $\ell = 1$, then $I_t(L)$ is generated by one indeterminate. In particular, it is prime and Cohen-Macaulay. Moreover $\tilde{\mathcal{L}} = \mathcal{L}$ and $\text{ht } I_t(L) = 1 = |\tilde{\mathcal{L}}|$.

We now prove the statement for a ladder \mathcal{L} with $|\mathcal{L}| = \ell$. By induction hypothesis, we may assume that the statement holds for any ladder with fewer than ℓ entries. If $t = (1, \dots, 1)$, then $I_t(L)$ is generated by ℓ indeterminates. In particular, it is prime and Cohen-Macaulay. Moreover $\tilde{\mathcal{L}} = \mathcal{L}$ and $\text{ht } I_t(L) = \ell = |\tilde{\mathcal{L}}|$.

Otherwise we have $t_k = \max\{t_1, \dots, t_s\} \geq 2$. By Remarks A.1 (2) we have $a_k > a_{k-1}$ and $b_k < b_{k+1}$. Let $t' = (t_1, \dots, t_{k-1}, t_k - 1, t_{k+1}, \dots, t_s)$ and let \mathcal{L}' be the ladder obtained from \mathcal{L} by removing the entries $(a_{k-1} + 1, b_k), \dots, (a_k - 1, b_k), (a_k, b_k), (a_k, b_k - 1), \dots, (a_k, b_{k+1} - 1)$. Let

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k - 1, b_k - 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

be the distinguished points of \mathcal{L}' . Notice that $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}'$. Since $|\mathcal{L}'| < \ell$, by induction hypothesis $I_{t'}(L')$ is Cohen-Macaulay and has $\text{ht } I_{t'}(L') = |\tilde{\mathcal{L}}'| = \ell$. Moreover, $I_{t'}(L')$ can be obtained from an ideal generated by indeterminates by a finite sequence of ascending elementary G-biliaisons. Let \mathcal{M} be the ladder obtained from \mathcal{L} by removing the entry (a_k, b_k) , and let

$$(a_1, b_1), \dots, (a_{k-1}, b_{k-1}), (a_k - 1, b_k), (a_k, b_k + 1), (a_{k+1}, b_{k+1}), \dots, (a_s, b_s)$$

be the distinguished points of \mathcal{M} . Let $u = (t_1, \dots, t_{k-1}, t_k, t_k, t_{k+1}, \dots, t_s)$ and let $I_u(M)$ be the corresponding ladder determinantal ideal. Notice that $\tilde{\mathcal{M}}$ is obtained from \mathcal{L} by removing the entry $(a_k - t_k + 1, b_k + t_k - 1)$. Since $|\mathcal{M}| = \ell - 1 < \ell$, by induction hypothesis $I_u(M)$ is prime and Cohen-Macaulay, and $\text{ht } I_u(M) = |\tilde{\mathcal{M}}| = \ell - 1$. As shown in [16, Theorem 2.1], $I_t(L)$ is obtained from $I_{t'}(L')$ by an ascending elementary G-biliasion of height 1 on $I_u(M)$. More precisely, let f_L be a t_k -minor of L which involves rows and columns a_k and b_k , and let $f_{L'}$ be the $(t_k - 1)$ -minor of the submatrix of L' obtained from the previous one by deleting rows and columns a_k and b_k . Then

$$I_t(L)/I_u(M) \cong I_{t'}(L') + I_u(M)/I_u(M)$$

and the isomorphism is given by multiplication by $f_L/f_{L'}$. By induction hypothesis, $I_u(M)$ is Cohen-Macaulay and prime, hence also generically Gorenstein. Therefore $I_t(L)$ is Cohen-Macaulay, since $I_{t'}(L')$ is. Moreover, $I_{t'}(L')$ is obtained from an ideal of linear forms by a sequence of ascending elementary G-biliasions. Therefore, the same holds for $I_t(L)$.

In order to prove that $I_t(L)$ is prime, we may assume that $t_j \geq 2$ for all $1 \leq j \leq s$. In fact, if $t_1 = 1$ let $\tau = (t_2, \dots, t_s)$ and $\mathcal{N} = \mathcal{L}_2 \cup \dots \cup \mathcal{L}_s$. Then $K[L]/I_t(L) \cong K[N]/I_\tau(N)$, hence $I_t(L)$ is prime if and only if $I_\tau(N)$ is. Similarly if $t_s = 1$. If $t_j = 1$ for some $1 < j < s$, let $\tau = (t_1, \dots, t_{j-1}, t_j + 1, t_{j+1}, \dots, t_s)$ and let \mathcal{N} be the ladder with distinguished points

$$(a_1, b_1), \dots, (a_{j-1}, b_{j-1}), (a_j + 1, b_j - 1), (a_{j+1}, b_{j+1}), \dots, (a_s, b_s).$$

Let $u = a_j + 1$ and $v = b_j - 1$. By Lemma A.4 there is an isomorphism

$$K[N]/I_\tau(N)[x_{u,v}^{-1}] \cong K[L]/I_t(L)[x_{a_{j-1}+1,v}, \dots, x_{a_j,v}, x_{u,b_j}, \dots, x_{u,b_{j+1}-1}, x_{u,v}^{-1}].$$

Since $x_{u,v} \notin L$, $I_t(L)$ is prime if $I_\tau(N)$ is.

Hence we may assume without loss of generality that $t_j \geq 2$ for all $1 \leq j \leq s$. We now show that $I_t(L)$ is a prime ideal. Let $(u, v) \in \mathcal{L}$ and let $A = K[L]/I_t(L)[x_{u,v}^{-1}]$. By induction hypothesis and by Lemma A.4 we have that A is an integral domain. Therefore, there is exactly one associated prime ideal $P_{u,v}$ of $I_t(L)$ such that $x_{u,v} \notin P_{u,v}$. If P is the only minimal associated prime of $I_t(L)$, then $I_t(L) = P$ is prime. Suppose by contradiction then that there is another associated prime Q of $I_t(L)$, $Q \neq P$. Since A is an integral domain, it must be $x_{u,v} \in Q$. Let $(i, j) \in \mathcal{L}$ such that $x_{i,j} \notin Q$. Existence of such an (i, j) follows from the observation that $I_t(L)$ is unmixed of height

$$\text{ht}(I_t(L)) = |\tilde{\mathcal{L}}| < |\mathcal{L}|.$$

Repeating the argument above for $x_{i,j}$ instead of $x_{u,v}$, one sees that $x_{i,j} \in P$. In particular, the image of $x_{i,j}$ in A is zero. However, from Lemma A.4 one can verify that the image of $x_{i,j}$ in A is different from zero. This yields a contradiction, hence $I_t(L)$ is prime. \square

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